Opportunistic Topological Interference Management

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***Abstract*—The topological interference management (TIM) problem studies the degrees of freedom (DoF) of partially-connected interference networks with no channel state information (CSI) at the transmitters except the network topology (i.e., partial connectivity). In this paper, we consider a variant of the TIM problem with uncertainty in network topology, where the channel state with partial connectivity is only known to belong to one of M states at the transmitters. In particular, the transmitter has access to all network topological information over M states, but is unaware of which state it falls in exactly for communication. The receiver at any state is aware of the exact state it falls in besides the network topologies of all states, and wish to recover as much highly-prioritized information at current state as possible. We formulate it as the opportunistic TIM problem with network uncertainty modeled by M state-varying network topologies. To adapt to network topology uncertainty and different message decoding priority, joint encoding and opportunistic decoding are enabled at the transmitters and receivers respectively. Specifically, being aware of all possible network topologies, each transmitter sends a signal jointly encoded from all messages desired over M states, say M distinct messages, and at a certain State m, Receiver k wishes to opportunistically decode the first $\pi_k(m) \in \{1, 2, \cdots, M\}$ higher-priority messages. Under this opportunistic TIM setting, we construct a multi-state conflict graph to capture the mutual conflict of messages over M states, and characterize the optimal DoF region of two classes of network topologies via polyhedral combinatorics. A remarkable fact is that, under an additional mild monotonous condition, the optimality conditions of orthogonal access and one-to-one interference alignment still apply to TIM with uncertainty in network topology.**

I. Introduction

The recent years have witnessed the growth of interference management (IM) techniques in the increasingly complex wireless communication environment involving connected and autonomous vehicles/drones/robots. The major challenge of IM, however, is the difficulty of acquiring accurate and timely channel state information (CSI) at the transmitters, especially when the communication environment exhibits some network uncertainty, due to e.g., vehicles' high-speed mobility, that it appears varying network connectivity patterns.

Recently, there is an emerging line of research to deal with interference by exploiting some coarse and easily attainable network topological information. It was motivated by the observation that certain communication links are unavoidably much weaker than others, owing to the fact that signal (e.g., mmWaves) power decays fast with distance and local shadowing effects. This suggests the use of a partially-connected bipartite graph to model, at least approximately, the network connectivity patterns. At first glance, such connectivity information is useless because it completely losses channel information of magnitude and phase. Surprisingly, in certain networks, IM techniques based on such topological CSIT perform as well as those using perfect CSIT. Thanks to the coarse CSI requirement and its remarkable performance, exploiting topological information for IM has attracted a lot of attention under the theme of topological interference management (TIM) [1].

There is a rich body of literature on TIM. The study of TIM was initiated by Jafar [1], in which a bridge between the TIM and index coding problems was built, together with the characterization of information-theoretic optimality for some fundamental IM techniques. The relevant follow-up works include multilevel TIM [2], TIM via TDMA [3], TIM with multiple antennas [4], with alternating connectivity [5]–[7], with reconfigurable antennas [8], and under constrained coherence patterns [9]–[11]. All of these frameworks, however, are dedicated to 'definite' networks in the sense that the network topology is pre-determined at the transmitters before communication. The TIM settings with uncertainty in network topology have not ever been formally investigated in the literature, while such uncertainty occurs in most emerging networks that involve high-mobility vehicles/drones.

To deal with such uncertainty in network topology, opportunistic communication is suggested to inspect the influence of 'indefinite' network topologies on the TIM problem. Broadly speaking, opportunistic communication is a way to adaptively utilize channel resources for efficient data transmission. The early study dates back to downlink multiuser scheduling in fast-fading wireless channels to ripen multi-user diversity gains [12]. In this work, we follow the formulations of opportunistic communications in [13]–[16], and focus on opportunistic decoding of degraded message sets at the receivers. Opportunistic decoding can be modeled as communicating several base and opportunistic message sets over multiple states, where the receivers should decode the base message set (cf. the basic communicate rate) regardless of the state, and opportunistically decode the additional message set (cf. the incremental communication rate) for a better channel state.

By modeling network topology uncertainty as a sequence of bipartite graphs capturing all possible network topologies over M states, we formulate the opportunistic TIM problem, in which the transmitters have no CSI available except for such graphs. Particularly, each transmitter sends a signal jointly encoded from all base and opportunistic messages, say M distinct messages, depending only on those graph information, without knowing with which graph the network is exactly associated. Using opportunistic decoding, Receiver k at State m is supposed to opportunistically decode the first $\pi_k(m) \in \{1, 2, \cdots, M\}$ higher-priority messages. To adapt to opportunistic decoding of degraded message sets, we consider a monotonous structure of network topologies, where the sets of transmitters connected to a given receiver over M states are *totally ordered*, i.e., one is a subset of another. In doing so, we are able to characterize the optimal DoF region of two classes of network topologies under the opportunistic TIM setting: the chordal networks studied in [3] and the half-rate feasible networks in [1]. Polyhedral combinatorics is the key of the proofs. Inspecting polyhedral structures (e.g., integrality and half-integrality of the extreme points) of the DoF region outer bounds, we design achievability using simple schemes (e.g., TDMA and interference alignment) for such extreme points, so as to achieve the entire region via time sharing. Remarkably, under the totally ordered condition, the structural properties that determine the information-theoretic optimality in TIM are also applicable to the opportunistic TIM problem. As a byproduct, the *symmetric DoF* result in the half-rate feasible networks in TIM [1] is also extended to *DoF region*, thanks to the half-integrality property of the DoF region.

Notations: For an integer N, $[N] \triangleq \{1, 2, \dots, N\}$. Given $n \in [N]$, we denote by $\{a_n\}_n$ a set of N elements, i.e., $\{a_n\}_n \triangleq \{a_1, a_2, \dots, a_N\}$, and similarly $\{a_{m,n}\}_{m,n}$ given $m \in [M]$ and $n \in [N]$ is a set with MN elements, i.e., $\{a_{m,n}\}_{m,n} \triangleq \{a_{1,1}, a_{1,2}, \dots, a_{1,N},\ a_{2,1}, \dots, a_{M,N}\}$. This rule of notations also applies to those with index sets at the superscripts or subscripts.

II. System Model

A. Topological Interference Management with States

Consider the K-user single-antenna partially-connected Gaussian interference network with M states. The received signal for Receiver k over the t-th channel use when the network falls into the m-th state is given by

$$Y_k^{[m]}(t) = \sum_{i \in \mathcal{T}_k^{[m]}} h_{ki}^{[m]} X_i(t) + Z_k^{[m]}(t),\ \forall k \in [K], \forall m \in [M]$$

where $h_{ki}^{[m]}$ is the complex channel coefficient from Transmitter i to Receiver k at the m-th state, and it keeps fixed at each state yet may be varying across states, $\mathcal{T}_k^{[m]}$ is the set of transmitters connected to Receiver k at State-m and can be distinct across states, $X_i(t)$ is the transmitted signal that depends only on the network topology $\{\mathcal{T}_k^{[m]}\}_{k,m}$ with average power constraint $\sum_{t=1}^n \mathbb{E}\left(\|X_i(t)\|^2\right) \leq nP$, and $Z_k^{[m]} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ is the normalized additive white Gaussian noise.

Throughout this paper, we assume that $\{\mathcal{T}_k^{[m]}\}_m$ is a *totally ordered set* for each k satisfying the properties of reflexivity, antisymmetry, transitivity, and comparability.[1] In particular, the comparability is such that, for any m_1 and m_2, either $\mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}$ or $\mathcal{T}_k^{[m_2]} \subseteq \mathcal{T}_k^{[m_1]}$. As a result, for every k, $\{\mathcal{T}_k^{[m]}\}_m$ can be totally ordered as

$$\mathcal{T}_k^{[m_{i_1}]} \subseteq \mathcal{T}_k^{[m_{i_2}]} \subseteq \cdots \subseteq \mathcal{T}_k^{[m_{i_M}]}. \tag{1}$$

Under the opportunistic TIM setting, it is assumed that each transmitter is aware of the network topology over the whole network across states (i.e., $\{\mathcal{T}_k^{[m]}\}_{m,k}$), yet has no knowledge of channel coefficients $h_{ki}^{[m]}$ or the network topology changing pattern. That is, the transmitters know all possible network topologies, but does not know which one the current channel falls into exactly. The receivers have perfect channel knowledge including the network topology at each state.

The network topology at State m refers to a bipartite graph $G^{[m]}$ with the transmitter set on one side and the receiver set on the other side, and the edge set $\{(j,k) : \forall j \in \mathcal{T}_k^{[m]}\}$. We use hereafter the set of bipartite graphs $\mathcal{G}$ to denote all the possible network topologies across M states, in which $G^{[m]} \in \mathcal{G}$ is a realization of the network topology at State m.

In what follows, similarly to [16], we define encoding and decoding functions for opportunistic communications.

Encoding: At each Transmitter i ($i \in [K]$), a set of independent messages $\{W_i^{[m]}\}_{m=1}^M$, uniformly chosen from the index set $\mathcal{W}_i^{[m]} \triangleq \{1, 2, \dots, \lceil 2^{nR_i^{[m]}} \rceil\}$, is jointly mapped to the codeword $\{X_i(t)\}_{t=1}^n \in \mathcal{X}_i^n$, given the set of all network topologies $\{G^{[m]}\}_m \in \mathcal{G}^M$. The codeword $\{X_i(t)\}_{t=1}^n$ is transmitted over n channel uses, and is subject to the average power constraint $\sum_{t=1}^n \mathbb{E}\left[|X_i(t)|^2\right] \leq nP$. For every $i \in [K]$, such a mapping can be described by a single encoding function

$$f_i : \textstyle\prod_{m=1}^M \mathcal{W}_i^{[m]} \mapsto \mathcal{X}_i^n, \tag{2}$$

where the codeword $\{X_i(t)\}_t$ does not depend on channel coefficients $\{h_{ki}^{[m]}\}_{k,i,m}$ but on network topologies $\{G^{[m]}\}_m$.

Decoding: At the m'-th state, the received signal $\{Y_k^{[m']}(t)\}_{t=1}^n \in \mathcal{Y}_k^{[m']}$ for Receiver k is used to estimate the basic and opportunistic messages $\{W_k^{[m]}\}_{m=1}^{\pi_k(m')}$, yielding $\{\hat{W}_k^{[m]}\}_{m=1}^{\pi_k(m')}$, given the knowledge of all network topologies and the perfect channel state information over M states. The number of the messages to be decoded by Receiver k at State m', $\pi_k(m') \in [M]$, is fixed *a priori* and globally known. Without loss of generality, we assume for all k

$$\pi_k(m_1) \geq \pi_k(m_2), \quad \text{if } \mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}. \tag{3}$$

Thus, the decoding function at the m'-th state (may be distinct across states) for Receiver k can be described by

$$g_k^{[m']} : \mathcal{Y}_k^{[m']} \mapsto \textstyle\prod_{m=1}^{\pi_k(m')} \mathcal{W}_k^{[m]}, \quad \forall m' \in [M]. \tag{4}$$

The average probability of error is defined as follows

$$P_e^{(n)} = \Pr\left(\bigcup_{m'=1}^M \left\{ \left(\{W_k^{[1:\pi_k(m')]}\}_k\right) \neq \left(\{\hat{W}_k^{[1:\pi_k(m')]}\}_k\right) \right\} \right).$$

A rate tuple $(\{R_k^{[m]}\}_{k,m})$ is said to be achievable if we have a set of encoding $\{f_i\}_i$ and decoding functions $\{g_k^{[m]}\}_{k,m}$ such

[1] For instance, the sets $\{1,2\}, \{1,2\}, \{1,2,3\}$ are totally ordered, while $\{1,2\}, \{2,3\}, \{1,2,3\}$ are not because $\{1,2\}$ and $\{2,3\}$ are not comparable.

that $P_e^{(n)} \to 0$ as $n \to \infty$. The capacity region $\mathcal{C}$ is the closure of the set of all achievable rate tuples. The degrees of freedom (DoF) region with respect to $(\{d_k^{[m]}\}_{k,m})$ is defined as follows.

$$\mathcal{D} \triangleq \Bigg\{ (\{d_k^{[m]}\}_{k,m}) \in \mathbb{R}_+^{MK} : d_k^{[m]} = \lim_{P\to\infty} \frac{R_k^{[m]}}{\log P}, \\ \forall k \in [K], m \in [M], (\{R_k^{[m]}\}_{k,m}) \in \mathcal{C} \Bigg\}.$$

B. Definitions

In what follows, some graph theoretic definitions are briefly recalled, and more details can be found in e.g., [17].

Both the directed graphs (digraphs) and its underlying undirected version (graphs) will be considered. We follow the standard notations in graph theory. The digraph $D = (V, A)$ consists of a vertex set V and an arc set A, where $A(D)$ denotes the arc set of D. An arc $(u, v) \in A$ with $u, v \in V$ is a directed edge from u to v. The incoming and outgoing neighborhood of $v \in V$ is the sets of vertices $\mathcal{N}^-(v) \triangleq \{u : (u, v) \in A\}$ and $\mathcal{N}^+(v) \triangleq \{u : (v, u) \in A\}$, respectively. The underlying undirected graph of D, usually denoted by $U(D) = (V, E)$, is such that $(u, v) \in E$ in $U(D)$ if and only if any of $(u, v) \in A$ and $(v, u) \in A$ exists in D. The complement of a graph $G = (V, E)$, denoted by $\bar{G} = (V, \bar{E})$, has the same vertex set V and $(u, v) \in \bar{E}(\bar{G})$ if and only if $(u, v) \notin E(G)$. In the undirected graphs, a chordless cycle with length n, denoted by C_n, is a closed loop of n vertices and edges without chord, i.e., $C_n = \{(v_1, v_2), (v_2, v_3), \dots, (v_n, v_1)\}$, where a chord is an edge that connects two non-adjacent vertices of a cycle. The hole is the chordless cycle C_n with $n > 4$, and the antihole $\bar{C}_n$ is its complement. The odd hole is a hole C_n with odd $n = 5, 7, \dots$, and the odd antihole is its complement. An undirected graph is perfect if and only if it does not contain odd holes or odd antiholes as induced subgraphs. The chordal bipartite graphs are a class of bipartite graphs that does not contain induced chordless cycles C_n with $n \geq 6$. As bipartite graphs only contain even cycles, chordal bipartite graphs either does not contain any cycles or contain only C_4.

A key step is the construction of a multi-state conflict digraph D over M states. We use G, D, and $U(D)$ to denote respectively the network topology, its conflict digraph, and the undirected version.

Definition 1 (Multi-state Conflict Digraph)**.** Let $D^{[m]} = (V^{[m]}, A^{[m]})$ be the single-state conflict digraph of the original network topology $G^{[m]}$ with regard to the message set $\{(W_k^{[1:\pi_k(m)]})\}_k$. The vertex $v_k^{[m]} \in V^{[m]}$ represents the set of desired messages $(W_k^{[1:\pi_k(m)]})$ by Receiver k, and an arc $(v_i^{[m]}, v_j^{[m]}) \in A^{[m]}$ exists if and only if Transmitter i is connected to Receiver j at State m with $i \neq j$ in $G^{[m]}$.

The multi-state conflict digraph $D = (V, A)$ for M states is constructed from $\{D^{[m]}\}_m$ such that

$$V = \cup_{m=1}^{M} V^{[m]} \tag{5}$$

$$A = \cup_{m_1=1}^{M} \cup_{m_2=1}^{M} A^{[(m_1,m_2)]}, \tag{6}$$

where for $v_i^{[m_1]} \in V^{[m_1]}$ and $v_j^{[m_2]} \in V^{[m_2]}$, $(v_i^{[m_1]}, v_j^{[m_2]}) \in A^{[(m_1,m_2)]}$ holds whenever $(v_i^{[m_2]}, v_j^{[m_2]}) \in A^{[m_2]}$ holds. Intuitively, if Transmitter i interferes Receiver j at one state, then Receiver j at *such* state will be always interfered by Transmitter i at *any* state. When $m_1 = m_2$, $A^{[(m_1,m_2)]}$ reduces to $A^{[m_2]}$. Note that $v_k^{[m_1]}$ and $v_k^{[m_2]}$ are not adjacent for any $m_1 \neq m_2$. For a concrete example, see Fig. 1(a).

For Receiver k at two different States m_1 and m_2, when $\mathcal{T}_k^{[m_1]} = \mathcal{T}_k^{[m_2]}$, we usually set $\pi_k(m_1) = \pi_k(m_2)$, where Receiver k is supposed to decode the same set of messages at both States m_1 and m_2. In this case, the corresponding vertices in D and $U(D)$ can be merged for simplicity. In particular, if two vertices $v_k^{[m_1]}$ and $v_k^{[m_2]}$ for Receiver k have identical incoming neighborhood $\mathcal{N}^-(v_k^{[m_1]}) = \mathcal{N}^-(v_k^{[m_2]})$ in D, then they can be merged as a single vertex. For such a vertex, the incoming neighborhood is still $\mathcal{N}^-(v_k^{[m_1]})$, whereas the outgoing neighborhood will be $\mathcal{N}^+(v_k^{[m_1]}) \cup \mathcal{N}^+(v_k^{[m_2]})$. An example will be shown in Fig. 3.

The multi-state conflict graph $U(D) = (V, E)$ is the underlying undirected graph of $D = (V, A)$ where $(v_i^{[m_1]}, v_j^{[m_2]}) \in E$ if and only if $(v_i^{[m_1]}, v_j^{[m_2]}) \in A$ and/or $(v_j^{[m_2]}, v_i^{[m_1]}) \in A$. Similarly, the undirected conflict graph of $G^{[m]}$ at State m can be denoted by $U(D^{[m]})$. For a concrete example, see Fig. 1(b).

Given such a construction, the opportunistic TIM problem over multiple states can be studied in a unique multi-state conflict graph, where some nice properties of single-state conflict graphs can be inherited by the multi-state ones.

Definition 2 (Auxiliary Network Topology)**.** The auxiliary network topology $G^{[\tilde{\boldsymbol{m}}]}$ with $\tilde{\boldsymbol{m}} = (m_1, m_2, \dots, m_K) \in [M]^K$ is a bipartite graph with the same transmitter and receiver sets, and the edge set $\{(j, k) : \forall j \in \mathcal{T}_k^{[m_k]}\}$ for all k. Roughly speaking, $G^{[\tilde{\boldsymbol{m}}]}$ is comprised of $\{\mathcal{T}_1^{[m_1]}, \mathcal{T}_2^{[m_2]}, \dots, \mathcal{T}_K^{[m_K]}\}$. When $m_1 = m_2 = \dots = m_K = m$, it reduces to the original network topology $G^{[m]}$ at State m.

For the 2-state network topologies in Fig. 1, the auxiliary network topologies are given in Fig. 2.

Definition 3 (All-chordal Networks)**.** The all-chordal networks are such that all original and auxiliary network topologies are chordal bipartite graphs.

Definition 4 (Internal Alignment Arc)**.** Given the multi-state conflict digraph D, $(v_i^{[m_i]}, v_j^{[m_j]}) \in A(D)$ is an internal alignment arc if the two vertices interfere some vertex $v_k^{[m_k]}$, i.e., $\exists\ k \neq i, j \in [K]$

$$(v_i^{[m_i]}, v_k^{[m_k]}) \in A(D) \quad \text{and} \quad (v_j^{[m_j]}, v_k^{[m_k]}) \in A(D). \tag{7}$$

Internal alignment is an inheritable property from single-state to multi-state conflict digraphs, in that i, j, k are distinct and $v_i^{[m_i]}$, $v_j^{[m_j]}$, and $v_k^{[m_k]}$ fall exactly into one original or auxiliary network. Intuitively, the corresponding messages associated with the internal alignment arc interfere one another and at the same time should be aligned because they interfere another message other than themselves. The existence of an internal

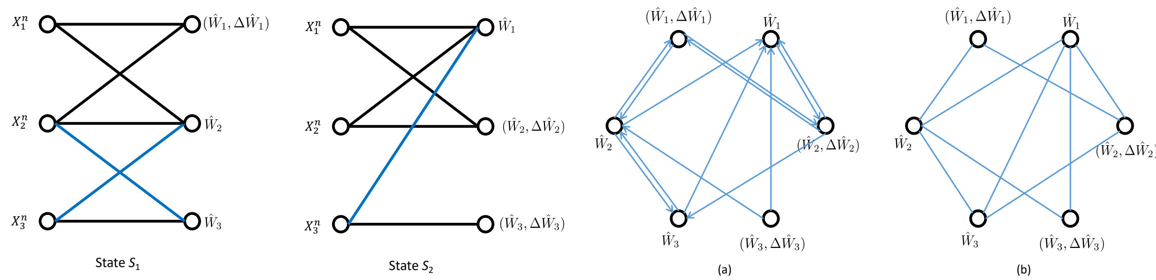

Fig. 1: (Left) The network topologies $G^{[1]}$ and $G^{[2]}$ at States S_1 and S_2, where the blue links indicate the state-varying connectivity. (Right) The multi-state conflict digraph D (a) over two states and its undirected version $U(D)$ (b), where the vertices represent the desired messages at different states and the arcs (edges) indicate the conflict of messages. For example, given that Transmitter 3 interferes Receiver 1 at State S_2, we have both arcs from $\hat{W}_3$ at State S_1 and $(\hat{W}_3, \Delta\hat{W}_3)$ at State S_2 to $\hat{W}_1$ at State S_2, i.e., $(v_3^{[1]}, v_1^{[2]}) \in A(D)$ and $(v_3^{[2]}, v_1^{[2]}) \in A(D)$.

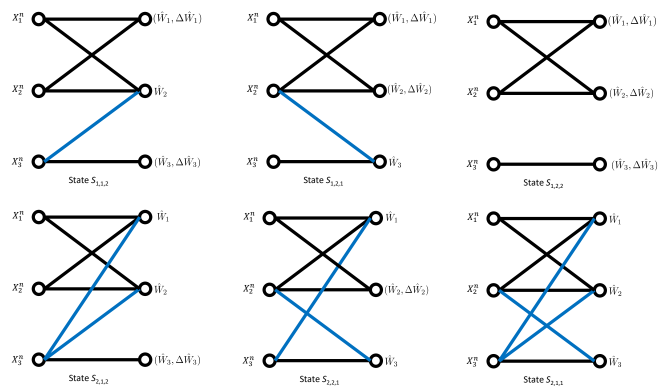

Fig. 2: The $2^3 - 2 = 6$ auxiliary network topologies, in each of which the transmit set of Receiver k is chosen from either $\mathcal{T}_k^{[1]}$ or $\mathcal{T}_k^{[2]}$ for all k. For instance, at State $S_{1,2,1}$, the transmit sets at three receivers are $\mathcal{T}_1^{[1]}$, $\mathcal{T}_2^{[2]}$, and $\mathcal{T}_3^{[1]}$, respectively.

alignment arc in conflict digraphs is equivalence to the existence of an internal conflict edge in alignment graphs defined in [1]. Thus, the half-rate feasible networks in [1] are a class of network topologies such that there does not exist any internal alignment arc in their conflict digraphs.

Definition 5 (All-half-rate-feasible Networks)**.** The all-half-rate-feasible networks are such that all original and auxiliary network topologies belong to the half-rate feasible networks. Equivalently, they refer to a class of network topologies whose multi-state conflict digraphs have no internal alignment arcs.

Definition 6 (Edge/Clique Inequalities)**.** For an undirected graph $G = (V, E)$, the collection of all edge inequalities is

$$\forall (i,j) \in E, \quad x_i + x_j \le 1, \tag{8}$$

and the collection of all clique inequality is

$$\forall Q \text{ is a clique in } G, \quad \sum_{i \in Q} x_i \le 1. \tag{9}$$

The cliques with $|Q| = 1$ and $|Q| = 2$ correspond respectively to the vertices and the edges, so clique inequalities include individual and edge inequalities as special cases. In addition, the non-negative individual inequalities refer to

$$\forall i \in V, \quad 0 \le x_i \le 1. \tag{10}$$

III. Main Results

In what follows, we consider the M-state K-user opportunistic TIM problem with original network topologies $\{G^{[m]}\}_m$ for which $\{\mathcal{T}_k^{[m]}\}_m$ are totally ordered sets for every k. The detailed proofs are relegated to Appendix.

Theorem 1. *For the opportunistic TIM problem, the optimal DoF region of the all-chordal networks is the collection of all non-negative individual and clique inequalities in the undirected multi-state conflict graph $U(D)$. Specifically, the optimal DoF region, which can be achieved by TDMA, includes all DoF tuples $(\{d_k^{[m]}\}_{k,m}) \in \mathbb{R}_+^{MK}$ satisfying*

$$\sum_{k=1}^{k'} \sum_{m=1}^{\pi_{i_k}(m_{i_k})} d_{i_k}^{[m]} \le 1, \text{ if } \{(v_{i_k}^{[m_{i_k}]})\}_{k=1}^{k'} \text{ forms a clique in } U(D)$$
$$\forall (i_1, i_2, \dots, i_{k'}) \subseteq [K], \ \forall (m_{i_1}, m_{i_2}, \dots, m_{i_{k'}}) \in [M]^{k'}, \tag{11}$$

where $(i_1, i_2, \dots, i_{k'})$ is any possible subset of $[K]$ with k' distinct elements, $[M]^{k'}$ is a set with cardinality $M^{k'}$ collecting all possible k'-ary tuples, each coordinate of which is from $[M]$, and $\pi_k(m) \in [M]$ is the number of desired messages by Receiver k at State m, which is globally known a priori.

Remark 1. The DoF region in (11) is integral, such that all extreme points of the polytope have binary-valued coordinates.

Example 1. Let us consider a 3-user opportunistic TIM problem with 2 states, denoted by S_1 and S_2, whose network topologies over two states are given in Fig. 1. The multi-state conflict (di)graphs are given in Fig. 1, and the auxiliary network topologies are given in Fig. 2. For the sake of notational clarity, we denote by W_k and ΔW_k respectively the basic and opportunistic messages. Thus, we let $d_k = d_k^{[1]}$ and $\Delta d_k = d_k^{[2]}$. The receiver sets $\{\mathcal{T}_k^{[m]}\}_{k,m}$ over two states are as follows

$$\mathcal{T}_1^{[1]} = \{1,2\}, \quad \mathcal{T}_2^{[1]} = \{1,2,3\}, \quad \mathcal{T}_3^{[1]} = \{2,3\} \tag{12a}$$
$$\mathcal{T}_1^{[2]} = \{1,2,3\}, \quad \mathcal{T}_2^{[2]} = \{1,2\}, \quad \mathcal{T}_3^{[2]} = \{3\} \tag{12b}$$

As a natural choice, we set

$$\pi_1(1) = 2, \quad \pi_2(1) = 1, \quad \pi_3(1) = 1 \tag{13a}$$
$$\pi_1(2) = 1, \quad \pi_2(2) = 2, \quad \pi_3(2) = 2 \tag{13b}$$

where $\pi_1(1) = 2$ as $\mathcal{T}_1^{[1]} \subset \mathcal{T}_1^{[2]}$ suggests that Receiver 1 at State 1 has a superior decoding capability with fewer interfering transmitters, and is supposed to decode more messages.

It can be checked that, all the original and auxiliary states are chordal bipartite, so Theorem 1 follows. Thus, according to (11), besides the non-negative individual inequalities, we have the optimal DoF region consisting of inequalities

$$d_1 + \Delta d_1 + d_2 + \Delta d_2 \leq 1 \quad (14a)$$
$$d_1 + d_2 + \Delta d_2 + d_3 \leq 1 \quad (14b)$$
$$d_1 + d_2 + d_3 + \Delta d_3 \leq 1. \quad (14c)$$

The achievability of the optimal DoF region can be verified by checking every extreme point. The nontrivial DoF tuples $(d_1, d_2, d_3, \Delta d_1, \Delta d_2, \Delta d_3)$ are the extreme points $(0, 0, 0, 1, 0, 1)$, $(0, 0, 1, 1, 0, 0)$, and $(0, 0, 0, 0, 1, 1)$. It is easy to verify the achievability using TDMA. For instance, to achieve the DoF tuple $(0, 0, 1, 1, 0, 0)$, we transmit W_3 and ΔW_1 at Transmitters 3 and 1 simultaneously. At State 1, we safely recover W_3 at Receiver 3 and ΔW_1 at Receiver 1 without any interference; At State 2, W_3 can still be recovered at Receiver 3. Note that at State 2, Receiver 1 is not required to decode any information because the rate of the base message W_1 is 0.

Theorem 2. *For the opportunistic TIM problem, the optimal DoF region of all-half-rate-feasible networks is the collection of all non-negative individual and edge inequalities in the undirected multi-state conflict graph. Specifically, the optimal DoF region, which can be achieved by interference alignment, includes all DoF tuples* $(\{d_k^{[m]}\}_{k,m}) \in \mathbb{R}_+^{MK}$ *satisfying*

$$\sum_{m=1}^{\pi_k(m')} d_k^{[m]} \leq 1, \quad \forall k \in K, \forall m' \in [M] \quad (15a)$$

$$\sum_{m=1}^{\pi_k(m_1)} d_k^{[m]} + \sum_{m=1}^{\pi_j(m_2)} d_j^{[m]} \leq 1, \quad \forall (v_k^{[m_1]}, v_j^{[m_2]}) \in E,$$
$$\forall m_1, m_2 \in [M], \forall k, j \in [K] \quad (15b)$$

where E is the edge set of the multi-state conflict graph $U(D)$.

Remark 2. The DoF region in (15) is half-integral, in the sense that the coordinates of all extreme points of the polytope take values from $\{0, 1, \frac{1}{2}\}$.

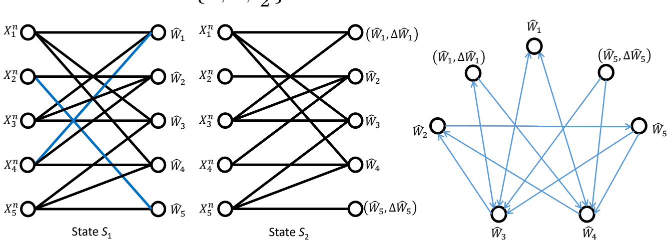

Fig. 3: The network topology with 2 states, and the multi-state conflict digraph D. All original and auxiliary states are half-rate feasible because no internal alignment arcs exist in the conflict digraph.

Example 2. We consider another example of a 5-user network with two states as in Fig. 3. According to the transmit set over two states, we set $\pi_k(1) = 1$ for all k, $\pi_1(2) = \pi_5(2) = 2$, and $\pi_2(2) = \pi_3(2) = \pi_4(2) = 1$. There does not exist internal alignment arc in the multi-state conflict digraph, so Theorem 2 applies, and the optimal DoF region consists of inequalities

$$d_1 + \Delta d_1 + d_3 \leq 1, \quad d_1 + \Delta d_1 + d_4 \leq 1 \quad (16a)$$
$$d_2 + d_3 \leq 1, \quad d_2 + d_4 \leq 1, \quad d_2 + d_5 \leq 1 \quad (16b)$$
$$d_3 + d_5 + \Delta d_5 \leq 1, \quad d_4 + d_5 + \Delta d_5 \leq 1 \quad (16c)$$

The nontrivial DoF tuples $(d_1, d_2, d_3, d_4, d_5, \Delta d_1, \Delta d_5)$ at the extreme points of the polytope are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)$, $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$, $(1, 1, 0, 0, 0, 0, 1)$, and $(0, 1, 0, 0, 0, 1, 1)$, and they can be achieved by carefully aligning interference.

IV. Conclusion

The topological interference management problem with uncertainty in network topology has been studied. With the aid of polyhedral combinatorics, the optimal DoF region of two classes of network topologies have been characterized.

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Appendix: Proof of Theorem 1

For notational simplicity, we define

$$d_{k,\mathrm{sum}}^{[m_k]} \triangleq \sum_{m=1}^{\pi_k(m_k)} d_k^{[m]} \tag{17}$$

as the sum DoF with respect to the desired basic and opportunisitc messages $\{W_k^{[m]}\}_{m=1}^{\pi_k(m_k)}$ for Receiver k at State m_k. In general, we introduce a linear transformation from the DoF tuple $(\{d_k^{[m]}\}_{k,m}) \in \mathbb{R}_+^{MK}$ to the sum DoF tuple $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k}) \in \mathbb{R}_+^{MK}$, i.e.,

$$f: (\{d_k^{[m]}\}_{k,m}) \mapsto (\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k}) \tag{18}$$

where such a mapping function f is surjective yet non-necessarily injective.

The proof of Theorem 1 is due to the following matching converse and achievablity. The converse proof will deal with DoF region with respect to $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k})$, whereas the achievability proof will be dedicated to $(\{d_k^{[m]}\}_{k,m})$. The optimality will be due to the above mapping f. For ease of notation, we denote by $\mathcal{P}^*$ the DoF region defined in (11).

A. Converse

To simplify the converse proof, we cast our problem to a set of regular TIM problems for which the optimal DoF regions have been characterized in [3] given the network topology is chordal bipartite. In doing so, we directly collect the clique inequalities therein to form our DoF region outer bound.

Lemma 1. *Any message set can be decoded in the M-state opportunistic TIM problem with network topologies $\{G^{[m]}\}_m$ if and only if the same message set can be decoded in every TIM instance with network topology $G^{[\tilde{\boldsymbol{m}}]}$, for all $\tilde{\boldsymbol{m}} \in [M]^K$.*

This lemma is analogous to the one for opportunistic TIN [16, Lemma 3], and thus the proof is omitted. In short, whether or not the messages can be decoded at a receiver is determined by the marginal distribution associated to this receiver if there is no receiver cooperation. Thus the same message set can be decoded in both the M-state opportunistic TIM and the M^K regular TIM instances as the receivers in the two scenarios see the same marginal channel transition probabilities.

Lemma 2. *(From [3, Theorem 1]) Consider a single state $\tilde{\boldsymbol{m}} = (m_1, m_2, \ldots, m_K) \in [M]^K$. If $G^{[\tilde{\boldsymbol{m}}]}$ is chordal bipartite, the achievable DoF tuple $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_k) \in \mathbb{R}_+^K$ in such a single-state TIM instance should satisfy*

$$\sum_{k=1}^{k'} d_{i_k,\mathrm{sum}}^{[m_{i_k}]} \leq 1, \text{ if } \{v_{i_k}^{[m_{i_k}]}\}_{k=1}^{k'} \text{ forms a clique in } U(D^{[\tilde{\boldsymbol{m}}]}) \quad \forall (i_1, i_2, \ldots, i_{k'}) \subseteq [K] \tag{19}$$

where $U(D^{[\tilde{\boldsymbol{m}}]})$ is the undirected single-state conflict graph of the network topology $G^{[\tilde{\boldsymbol{m}}]}$.

For each $\tilde{\boldsymbol{m}} \in [M]^K$, the single-state conflict digraph $D^{[\tilde{\boldsymbol{m}}]} = (V^{[\tilde{\boldsymbol{m}}]}, A^{[\tilde{\boldsymbol{m}}]})$ is the induced sub-digraphs of the multi-state conflict digraph $D = (V, A)$. Thus, an induced subgraph is a clique in $U(D^{[\boldsymbol{m}]})$ if and only if it is a clique in $U(D)$. As such, the DoF region outer bound of M^K regular TIM instances can be written as follows.

Lemma 3. *For the M^K regular TIM instances, if for every $\tilde{\boldsymbol{m}} \in [M]^K$, $G^{[\tilde{\boldsymbol{m}}]}$ is chordal bipartite, the achievable DoF tuple $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k}) \in \mathbb{R}_+^{MK}$ should satisfy*

$$\sum_{k=1}^{k'} d_{i_k,\mathrm{sum}}^{[m_{i_k}]} \leq 1, \text{ if } \{v_{i_k}^{[m_{i_k}]}\}_{k=1}^{k'} \text{ forms a clique in } U(D) \quad \forall (i_1, i_2, \ldots, i_{k'}) \subseteq [K], \quad \forall (m_{i_1}, m_{i_2}, \ldots, m_{i_{k'}}) \in [M]^{k'} \tag{20}$$

where $U(D)$ is the multi-state conflict graph.

Lemma 3 says, the collection of the clique inequalities from $U(D^{[\tilde{\boldsymbol{m}}]})$ can be directly generated from $U(D)$.

By Lemmas 1 – 3, we conclude that the DoF region outer bound of the M-state opportunistic TIM problem for the all-chordal networks can be expressed as the collection of all clique inequalities in (20) of M^K regular TIM instances, each of which has chordal network topology.

We hereafter denote by $\mathcal{P}'$ the DoF region outer bound in (20), and let $(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{k,m_k})$ denote its extreme points. Noting that (20) and (11) are essentially identical, we have

$$\mathcal{P}^* = \mathcal{P}' = \mathrm{conv}(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{k,m_k}), \tag{21}$$

because a polytope can be represented by the convex hull of its extreme points [18, Theorem 5.10].

B. Achievability

We first study the integrality of extreme points $d_{k,\mathrm{sum}}^{*[m_k]}$ of $\mathcal{P}'$ for the all-chordal networks, and then translate it to the DoF tuples $\{d_k^{[m]}\}_{k,m}$ according to the mapping f. TDMA-based achievability schemes are then designed for the integral DoF tuple $\{d_k^{[m]}\}_{k,m}$, and using time sharing we generate an achievable DoF region $\mathcal{P} = f(\mathrm{conv}(\{d_k^{[m]}\}_{k,m}))$. Finally, we close the gap between the converse and the achievablility by showing that $\mathcal{P}^* = \mathcal{P}' = \mathcal{P}$.

Lemma 4. *(From [3, Lemma 2]) Given the network topology $G^{[\tilde{\boldsymbol{m}}]}$ ($\tilde{\boldsymbol{m}} \in [M]^K$), if it is chordal bipartite, then the single-state conflict graph $U(D^{[\tilde{\boldsymbol{m}}]})$ is perfect.*

This lemma is a special case of Lemma 2 in [3] with the interference message setting. It says, if $G^{[\tilde{\boldsymbol{m}}]}$ is chordal bipartite, its all-unicast message conflict graph (i.e., the square of line graph of $G^{[\tilde{\boldsymbol{m}}]}$) is perfect. The single-state conflict graph $U(D^{[\tilde{\boldsymbol{m}}]})$ is an induced subgraph of such an all-unicast message conflict graph, which is induced by the interference message set. Since the induced subgraphs of perfect graphs are still perfect, we conclude that $U(D^{[\tilde{\boldsymbol{m}}]})$ is also perfect.

Lemma 5. *Given the single-state conflict digraphs $D^{[\tilde{\boldsymbol{m}}]}$ and the multi-state one D, if*

- *for every $k \in [K]$, $\{\mathcal{T}_k^{[m]}\}_m$ is a totally ordered set , and*
- *for every $\tilde{\boldsymbol{m}} \in [M]^K$, $U(D^{[\tilde{\boldsymbol{m}}]})$ is a perfect graph,*

then the underlying undirected graph $U(D)$ is perfect.

In this lemma, both conditions should be satisfied, otherwise $U(D)$ is not necessarily perfect. While the second condition is straightforward, the first condition on totally ordered sets is not obvious. When the first condition is not satisfied, an example in Fig. 4 shows that $U(D)$ is not a perfect graph.

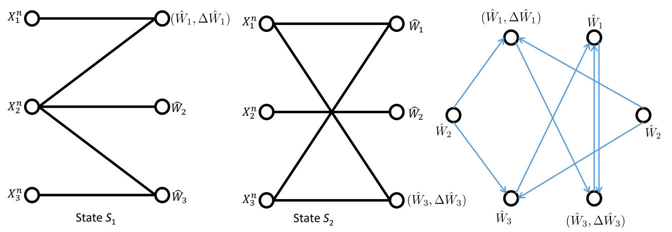

Fig. 4: An example showing that the totally ordered set condition is crucial. All original and auxiliary network topologies are chordal bipartite, yet the transmit sets are not totally ordered at Receivers 1 and 3. As a sequel, the multi-state conflict graph $U(D)$ is not perfect, because there exists an odd hole C_5.

Lemma 6. *(From [19], [21, Ch. 65]) If the multi-state conflict graph $U(D)$ is perfect, the polytope defined by its clique inequalities has only binary-valued extreme points.*

By Lemmas 3 and 6, we conclude that $\mathcal{P}'$ has only binary-valued extreme points with respect to $(\{d_{k,\text{sum}}^{[m_k]}\}_{k,m_k}) \in \mathbb{R}_+^{MK}$. The following lemma shows that such integrality can be inherited by the corresponding DoF tuple $(\{d_k^{[m]}\}_{k,m})$.

Lemma 7. *Given* (17)*, if $\{\mathcal{T}_k^{[m]}\}_m$ is totally ordered, the binary-valued sum DoF tuple $(\{d_{k,\text{sum}}^{*[m_k]}\}_{k,m_k})$ leads to the corresponding binary-valued DoF tuple $(\{d_k^{[m]}\}_{k,m})$.*

The following lemma shows the achievability of TDMA with respect to binary-valued $(\{d_k^{[m]}\}_{k,m}) \in \mathbb{R}_+^{MK}$.

Lemma 8. *The binary-valued DoF tuple $(\{d_k^{[m]}\}_{k,m}) \in \mathbb{R}_+^{MK}$ induced by $(\{d_{k,\text{sum}}^{*[m_k]}\}_{k,m_k})$ can be achieved by TDMA.*

The coordinates of the binary tuple $(\{d_k^{[m_k]}\}_{k,m})$ indicate if the corresponding messages are active (i.e., $d_k^{[m]} = 1$) or inactive (i.e., $d_k^{[m]} = 0$), where such an on-off decision corresponds to a TDMA link scheduling.

By time sharing among these achievable DoF tuples $(\{d_k^{[m]}\}_{k,m})$, any tuple in the convex hull $\text{conv}(\{d_k^{[m]}\}_{k,m})$ is achievable by TDMA. As f is a linear transformation, the DoF region

$$\mathcal{P} = f(\text{conv}(\{d_k^{[m]}\}_{k,m})) \tag{22}$$

is also achievable.

Lemma 9. *Given the linear transformation* (18)*, we have*

$$\mathcal{P}' = \text{conv}(\{d_{k,\text{sum}}^{*[m_k]}\}_{k,m_k}) \subseteq f(\text{conv}(\{d_k^{[m]}\}_{k,m})) = \mathcal{P}. \tag{23}$$

The intuition behind this lemma is that, given a mapping, the extreme points of the image (cf. codomain) of a compact convex set (cf. domain) are the subset of the images of extreme points of this compact convex set, and thus the relation of their convex hulls follows. When f is both injective and surjective, two polytopes are identical. Given the totally ordered set condition, the mapping f is in fact both injective and surjective.

Given the fact that $\mathcal{P} \subseteq \mathcal{P}^* = \mathcal{P}'$ implied by the converse, we conclude that $\mathcal{P} = \mathcal{P}' = \mathcal{P}^*$. This completes the optimality proof of Theorem 1.

C. Proofs of Key Lemmas

According to Definition 1 of the multi-state conflict digraph, we have the following two observations.

- **O1**: Given two vertices $v_k^{[m_1]}$ and $v_j^{[m_2]}$ with $k \neq j$, if $(v_k^{[m_1]}, v_j^{[m_2]}) \in A$, then for all $m \in [M]$, there must be $(v_k^{[m]}, v_j^{[m_2]}) \in A$. With respect to the original and auxiliary network topologies, if Transmitter k at *some* state interferes Receiver j at State-m_2, then Transmitter k at *all* M states will interfere Receiver j at State m_2.
- **O2**: Given two vertices $v_k^{[m_1]}$ and $v_k^{[m_2]}$, if $\mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}$, then for all $v_j^{[m_1]}$ such that $(v_j^{[m_1]}, v_k^{[m_1]}) \in A$, there must be $(v_j^{[m_1]}, v_k^{[m_2]}) \in A$. Equivalently, if Transmitter j interferes Receiver k at State-m_1, it will also interfere Receiver k at all other lower-order states that are supposed to have more interference presented.

Due to **O1** and **O2**, the multi-state conflict digraph has the following useful structural property.

- **Property**: Given a clique Q in $U(D)$ involving a vertex $v_k^{[m_1]}$, if $\mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}$, then $v_k^{[m_2]} \cup Q \backslash v_k^{[m_1]}$ also from a clique in $U(D)$.

That is because, for any vertex $u \in Q \backslash v_k^{[m_1]}$, there must be either $(u, v_k^{[m_1]}) \in A$ or $(v_k^{[m_1]}, u) \in A$, or both. If $(v_k^{[m_1]}, u) \in A$, due to **O1**, then $(v_k^{[m_2]}, u) \in A$. If $(u, v_k^{[m_1]}) \in A$, due to **O2**, then $(u, v_k^{[m_2]}) \in A$.

Let us proceed to prove the key lemmas in the previous subsection.

1) Proof of Lemma 3: We first show that, for each $\tilde{\boldsymbol{m}} \in [M]^K$ the single-state conflict digraph $D^{[\tilde{\boldsymbol{m}}]} = (V^{[\tilde{\boldsymbol{m}}]}, A^{[\tilde{\boldsymbol{m}}]})$ is the induced sub-digraphs of the multi-state conflict digraph $D = (V, A)$.

As the vertex set V is the collection of the vertex sets $V^{[\tilde{\boldsymbol{m}}]}$ for all $\tilde{\boldsymbol{m}} \in [M]^K$, we only need to show that, for any two vertices $v_k^{[m_{i_k}]}, v_j^{[m_{i_j}]} \in V$, $(v_k^{[m_{i_k}]}, v_j^{[m_{i_j}]}) \in A$ if and only if for all $\tilde{\boldsymbol{m}}$ subject to $[\tilde{\boldsymbol{m}}]_k = m_{i_k}$ and $[\tilde{\boldsymbol{m}}]_j = m_{i_j}$, there must be $(v_k^{[m_{i_k}]}, v_j^{[m_{i_j}]}) \in A^{[\tilde{\boldsymbol{m}}]}$. In other words, it says for any two receivers $k \neq j$ at any two states m_{i_k} and m_{i_j}, the arc connectivity maintains in all the single-state and multi-state conflict digraphs.

To this end, we proceed in the following way. According to the construction of the multi-state conflict digraph, $(v_k^{[m_{i_k}]}, v_j^{[m_{i_j}]}) \in A$ if and only if $(v_k^{[m_{i_k}]}, v_j^{[m_{i_j}]}) \in A_j^{[m_{i_j}]} \subseteq A^{[m_{i_j}]}$, where we use the arc set $A_j^{[m_{i_j}]}$ to specify all arcs coming to the vertex $v_j^{[m_{i_j}]}$. Due to the fact that $[\tilde{\boldsymbol{m}}]_j = m_{i_j}$, we have $A_j^{[m_{i_j}]} \subseteq A^{[\tilde{\boldsymbol{m}}]}$. As such, the arc connectivity in each single-state conflict digraph maintains in the multi-state conflict graph, and thus $D^{[\tilde{\boldsymbol{m}}]}$ is an induced sub-digraph of D.

Second, the underlying undirected graph maintains the conflicting structure of the digraph. That is, any two vertices $v_k^{[m_{i_k}]}$ and $v_j^{[m_{i_j}]}$ are conflicting if and only if they are conflicting at some original or auxiliary state $\tilde{\boldsymbol{m}}$ subject to $[\tilde{\boldsymbol{m}}]_k = m_{i_k}$ and $[\tilde{\boldsymbol{m}}]_j = m_{i_j}$. This conflict is also inherited by the multi-state conflict graph.

Thus, we conclude a clique in $U(D^{[\boldsymbol{m}]})$ if and only if it is in $U(D)$. Thus, the clique inequalities come from all the single-state conflict graphs $U(D^{[\boldsymbol{m}]})$ will be still valid to describe the clique inequalities in $U(D)$. This completes the proof.

2) Proof of Lemma 5: We prove this lemma by contradictions, that if $U(D)$ is not a perfect graph, then both conditions can not be satisfied at the same time.

Suppose $U(D) = (V, E)$ is not a perfect graph. Thus it may contain odd holes or odd antiholes as induced subgraphs, where $U(D)$ is the underlying undirected graph of $D = (V, A)$.

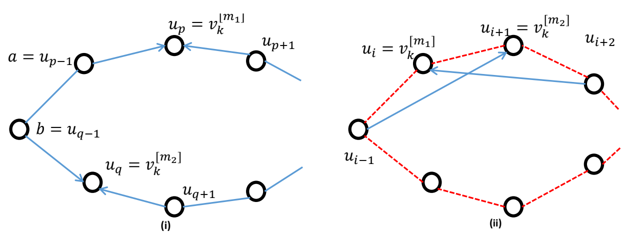

Fig. 5: (i) An odd holes in $U(D)$ and its possible arcs in D. (ii) An odd antihole in $U(D)$, which is represented as an odd hole in $\overline{U(D)}$ marked in dashed red line. The possible arc connectivity with possible vertices a and b in D is also marked using blue arrows.

- Suppose $U(D)$ contains an odd hole C_n where $n = 5, 7, 9, \ldots$ as an induced subgraph. Because $U(D^{[\tilde{\boldsymbol{m}}]})$ is perfect for every $\tilde{\boldsymbol{m}} \in [M]^K$, so C_n should not be a subgraph of any single-state conflict graph $U(D^{[\tilde{\boldsymbol{m}}]})$. As such, there exist two non-adjacent vertices $v_k^{[m_1]}$ and $v_k^{[m_2]}$ with $m_1 \neq m_2$ in C_n corresponding to the same receiver k at different states m_1 and m_2, because otherwise C_n should be contained in a single-state conflict graph $U(D^{[\tilde{\boldsymbol{m}}]})$ for some $\tilde{\boldsymbol{m}}$. We label the vertices of C_n by $u_1, u_2, \ldots, u_n$, so that $C_n = \{(u_1, u_2), (u_2, u_3), \ldots, (u_{n-1}, u_n), (u_n, u_1)\}$ and there are no other edges beyond these, because it is a chordless cycle. Let us label $u_p = v_k^{[m_1]}$ and $u_q = v_k^{[m_2]}$. Because of $n \geq 5$, there exist at least two vertices from u_p to u_q in C_n clockwise or anti-clockwise. There must exist two distinct vertices a and b chosen respectively from $\{u_{p-1}, u_{p+1}\}$ and $\{u_{q-1}, u_{q+1}\}$ in the digraph D such that

$$(a, v_k^{[m_1]}) \in A(D), \quad (v_k^{[m_1]}, a) \notin A(D) \tag{24a}$$
$$(b, v_k^{[m_2]}) \in A(D), \quad (v_k^{[m_2]}, b) \notin A(D) \tag{24b}$$
$$(a, v_k^{[m_2]}) \notin A(D), \quad (v_k^{[m_2]}, a) \notin A(D) \tag{24c}$$
$$(b, v_k^{[m_1]}) \notin A(D), \quad (v_k^{[m_1]}, b) \notin A(D) \tag{24d}$$

  where the last two conditions indicate that a is not adjacent to $v_k^{[m_2]}$ and b is not adjacent to $v_k^{[m_1]}$ because $n \geq 5$. The reasons are as follows. Since a is adjacent to $v_k^{[m_1]}$, at least one of $(a, v_k^{[m_1]})$ and $(v_k^{[m_1]}, a)$ is an arc in D. Suppose $(v_k^{[m_1]}, a) \in A(D)$, we have $(v_k^{[m_2]}, a) \in A(D)$ as well according to **O1**. This contradicts the fact that a and $v_k^{[m_2]}$ are not adjacent. So, we conclude that $(a, v_k^{[m_1]}) \in A(D)$ and $(v_k^{[m_1]}, a) \notin A(D)$ as in (24a). The similar argument applies to (24b) as well. The possible arc connectivity with a possible assignment of vertices a and b is shown in Fig. 5(i). As the in-neighborhood of a vertex $v_k^{[m]}$ in D indicates the transmit set $\mathcal{T}_k^{[m]}$, it follows from (24) that $\mathcal{T}_k^{[m_1]}$ and $\mathcal{T}_k^{[m_2]}$ are not comparable because $\mathcal{T}_k^{[m_1]} \nsubseteq \mathcal{T}_k^{[m_2]}$ and $\mathcal{T}_k^{[m_2]} \nsubseteq \mathcal{T}_k^{[m_1]}$, which contradicts the condition that $\{\mathcal{T}_k^{[m]}\}_m$ is a totally ordered set.
- Suppose $U(D)$ contains an odd antihole $\overline{C_n}$ where $n = 5, 7, 9, \ldots$ as an induced subgraph. This is equivalent to that $\overline{U(D)}$ contains an odd hole C_n as an induced subgraph. We label this hole as in Fig. 5(ii). There exist two adjacent vertices u_i and u_{i+1} belonging to C_n in $\overline{U(D)}$, such that $u_i = v_k^{[m_1]}$ and $u_{i+1} = v_k^{[m_2]}$ are associated with the same receiver at different states, because otherwise if all vertices in $\overline{U(D)}$ are from the different receivers, then the odd hole $\overline{U(D)}$ is an induced subgraph of some single-state conflict graph $U(D^{[\tilde{\boldsymbol{m}}]})$, which contradicts the condition that every $U(D^{[\tilde{\boldsymbol{m}}]})$ is perfect. Still, there exist two distinct vertices u_{i-1} and u_{i+2} such that

$$(u_{i-1}, v_k^{[m_2]}) \in A(D), \ (v_k^{[m_2]}, u_{i-1}) \notin A(D), \tag{25a}$$
$$(u_{i+2}, v_k^{[m_1]}) \in A(D), \ (v_k^{[m_1]}, u_{i+2}) \notin A(D), \tag{25b}$$
$$(u_{i-1}, v_k^{[m_1]}) \notin A(D), \ (v_k^{[m_1]}, u_{i-1}) \notin A(D), \tag{25c}$$
$$(u_{i+2}, v_k^{[m_2]}) \notin A(D), \ (v_k^{[m_2]}, u_{i+2}) \notin A(D), \tag{25d}$$

  where the last two conditions are due to the fact that $\overline{C_n}$ is an antihole. Further, $(v_k^{[m_2]}, u_{i+2}) \notin A(D)$ implies $(v_k^{[m_1]}, u_{i+2}) \notin A(D)$ and $(v_k^{[m_1]}, u_{i-1}) \notin A(D)$ implies $(v_k^{[m_2]}, u_{i-1}) \notin A(D)$ according to **O1**. At the same time, $(u_{i-1}, v_k^{[m_2]}) \in A(D)$ and $(u_{i+2}, v_k^{[m_1]}) \in A(D)$ should hold because otherwise there will exist chords $(u_{i-1}, v_k^{[m_2]})$ and $(u_{i+2}, v_k^{[m_1]})$ in $\overline{U(D)}$, which contradicts the fact that C_n is chordless. The possible arc connectivity is shown in Fig. 5(ii). This results in that the in-neighborhood of the vertices $v_k^{[m_1]}$ and $v_k^{[m_2]}$ in D, corresponding to the transmit sets $\mathcal{T}_k^{[m_1]}$ and $\mathcal{T}_k^{[m_2]}$ respectively, are not comparable, which contradicts the condition that $\{\mathcal{T}_k^{[m]}\}_m$ is a totally ordered set.

By far, given that all single-state conflict graphs are perfect, if $U(D)$ contains odd holes or odd antiholes, the transmit sets of a receiver across all states are not totally ordered. By contra-position, we conclude that given all perfect single-state conflict graphs, if the transmit sets are totally ordered, $U(D)$ should be also perfect. This completes the proof.

3) Proof of Lemma 7: Before presenting the proof, we first highlight a property of the extreme points of the polyhedron, which is imposed by the totally ordered network connectivity.

Lemma 10. *Given any extreme point* $(\{d_{k,\text{sum}}^{*[m_k]}\}_{k,m_k})$ *of the DoF region defined by* (11)*, if* $\mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}$ *we have* $d_{k,\text{sum}}^{*[m_1]} \geq d_{k,\text{sum}}^{*[m_2]}$*.*

Proof. The proof is due to an observation in polyhedral combinatorics. Consider a polytope defined by a set of linear inequalities with respect to $(\boldsymbol{x}, y, z)$. If for any linear inequality $\boldsymbol{a}^\intercal \boldsymbol{x} + y \leq b$ with respect to y, there *always* exists another linear inequality $\boldsymbol{a}^\intercal \boldsymbol{x} + z \leq b$ with respect to z, then for any extreme point $(\boldsymbol{x}^*, y^*, z^*)$ of the polytope we have $y^* \geq z^*$. It is because compared to y, z may be involved in more constraints in addition to those y involves, so that y dominates z.

The DoF region in (11) is defined by clique inequalities. According to **Property**, given $\mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}$, if there exists a clique Q involves $v_k^{[m_1]}$, the vertices $v_k^{[m_2]} \cup Q \backslash v_k^{[m_1]}$ must form a clique as well. Thus, for the clique inequality of Q involving $d_{k,\text{sum}}^{[m_1]}$,

$$\sum_{j \in Q \backslash v_k^{[m_1]}} d_j + d_{k,\text{sum}}^{[m_1]} \leq 1$$

there always exists another clique inequality by replacing $d_{k,\text{sum}}^{[m_1]}$ with $d_{k,\text{sum}}^{[m_2]}$, i.e.,

$$\sum_{j \in Q \backslash v_k^{[m_1]}} d_j + d_{k,\text{sum}}^{[m_2]} \leq 1.$$

Based on the above observation, we conclude that $d_{k,\text{sum}}^{*[m_1]} \geq d_{k,\text{sum}}^{*[m_2]}$, which completes the proof. □

Then, we proceed to prove Lemma 7. For ease of presentation, we rearrange $d_{k,\text{sum}}^{*[m_k]}$ with respect to m_k according to the order of comparability of $\{\mathcal{T}_k^{[m_k]}\}_{m_k}$ so that both $\{\pi_k(m)\}_m$ and $d_{k,\text{sum}}^{*[m_k]}$ are in ascending orders.

According to the definition of decoding, it is natural to restrict $\{\pi_k(m)\}_m$ to be a set of *consecutive* positive integers starting from 1. The repeated integers are allowed, which implies that the same set of messages are desired by several states. For instance, a valid set of $\{\pi_k(m)\}_m$ after rearrangement can be $\{1, 1, 2, 3, 4, \dots\}$, meaning that the same set of messages are desired by the first two states. On the other hand, the nonconsecutive integers are not allowed. For example, $\{1, 2, 4, 5, 6, \dots\}$ is not valid for $\{\pi_k(m)\}_m$ because it is unnecessary to request two additional messages from the second to the third state, while the two messages can be seen as a single one.

For ease of presentation, we assume there are M distinct messages over M states and $\{\pi_k(m)\}_m = [M]$. When there are repeated integers in $\{\pi_k(m)\}_m$, we add virtual messages with zero DoF. In the above example with repeated $\pi_k(1) = \pi_k(2) = 1$, we set $d_k^{[2]} = 0$. Let us denote by $\boldsymbol{d}_k = (\{d_k^{[m]}\}_m)^\intercal \in [0,1]^{M \times 1}$ and $\boldsymbol{d}_{k,\text{sum}}^* = (\{d_{k,\text{sum}}^{*[m]}\}_m)^\intercal \in \{0,1\}^{M \times 1}$. From (17), we have a system of linear equations

$$\boldsymbol{A}_k \boldsymbol{d}_k = \boldsymbol{d}_{k,\text{sum}}^* \tag{26}$$

where $\boldsymbol{A}_k$ is a lower triangular matrix. Due to Lemma 10, $\boldsymbol{d}_{k,\text{sum}}^*$ is not decreasing as $\pi_k(m)$ increases. If there are repeated integers in $\{\pi_k(m)\}_m$, $\boldsymbol{A}_k$ can be constructed by removing the columns corresponding to zero DoF in $\boldsymbol{d}_k$. For an exemplary choice of $\{\pi_k(m)\}_m = \{1, 1, 2, 3, 4, 4, 5\}$, $\boldsymbol{A}_k$ can be given as below

$$\boldsymbol{A}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{27}$$

By removing the redundant equations and performing one-step Gaussian elimination over the rows in $\boldsymbol{A}_k$, we have

$$\boldsymbol{A}'_k \boldsymbol{d}_k = \boldsymbol{d}'^*_{k,\text{sum}} \tag{28}$$

where one-step Gaussian elimination over rows is to subtract the $(i-1)$-th row from the i-th row in $\boldsymbol{A}_k$ for $i = \{M, M-1, \dots, 2\}$. After such operations, $\boldsymbol{A}'_k$ becomes an identity matrix, and $\boldsymbol{d}'^*_{k,\text{sum}}$ is still a binary-valued vector as $\boldsymbol{d}^*_{k,\text{sum}}$ is a non-decreasing binary-valued vector. So, we have $\boldsymbol{d}_k = \boldsymbol{d}'^*_{k,\text{sum}}$ which is binary-valued. This completes the proof.

4) Proof of Lemma 8: Given a binary-valued DoF tuple $(\{d_k^{[m]}\}_{k,m})$, we reshape it as a $K \times M$ binary matrix $\boldsymbol{D}$ with $[\boldsymbol{D}]_{km} = d_k^{[m]}$, where the k-th row corresponds to Receiver k, and the m-th column corresponds to additional messages that can be decoded at State m. According to the multi-state conflict graph, we have the following observations:

- Each row of $\boldsymbol{D}$ has at most one '1' which means every receiver can take at most one message at certain state over all M states.
- In m-th column of $\boldsymbol{D}$, the corresponding vertices of the nonzero elements form an independent set in $U(D^{[m]})$ and therefore in $U(D)$.
- Given any $m' \in [M]$, the non-zero elements in the first $\pi_k(m')$ columns fall into a certain auxiliary network, say $G^{[\tilde{\boldsymbol{m}}]}$, and the corresponding vertices form an independent set in $U(D^{[\tilde{\boldsymbol{m}}]})$ and therefore in $U(D)$.

Given a binary-valued DoF tuple $(\{d_k^{[m]}\}_{k,m})$, the messages with non-zero value (i.e., $d_k^{[m]} = 1$) in $\boldsymbol{D}$ will be transmitted simultaneously. For instance, $[\boldsymbol{D}]_{km} = 1$ indicates Transmitter k only sends the message $W_k^{[m]}$, and makes other messages dumb. At the receiver side, any receiver is able to recover exactly one desired message successfully, because no interference conflict exists. For instance, Receiver k at State m' aims to decode the message $W_k^{[m]}$ where $m \leq \pi_k(m')$, yielding $d_k^{[m]} = 1$. According to the above observations, any transmitter carrying non-zero DoF messages will be not seen by Receiver k at State m'; Otherwise, there should exist an conflict edge between them, which contradicts the definition of independent set. This means it takes exactly one time slot to transmit one message, and the desired message $W_k^{[m]}$ is decodable at Receiver k.

Hence, the binary-valued DoF tuple $(\{d_k^{[m]}\}_{k,m})$ is achievable by TDMA at the receivers. This completes the proof.

5) Proof of Lemma 9: In general, the lemma holds even for affine transformation, of which linear transformation is a special case. Given an affine transformation $f : X \mapsto Y$, we need to prove $\mathrm{conv}(f(X)) \subseteq f(\mathrm{conv}(X))$.

First, without loss of generality, we assume $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}$. Given $\boldsymbol{x}_j \in X$ and $\{\theta_j\}_j$ subject to $\sum_j \theta_j = 1$, we have

$$f\big(\sum_j \theta_j \boldsymbol{x}_j\big) = \boldsymbol{A}\sum_j \theta_j \boldsymbol{x}_j + \boldsymbol{b} \tag{29}$$

$$= \sum_j \theta_j(\boldsymbol{A}\boldsymbol{x}_j + \boldsymbol{b}) = \sum_j \theta_j f(\boldsymbol{x}_j) \tag{30}$$

Second, given $\boldsymbol{y} \in \mathrm{conv}(f(X))$, it implies there exist $\{\delta_j\}_j$ with $\sum_j \delta_j = 1$ such that $\boldsymbol{y} = \sum_j \delta_j f(\boldsymbol{z}_j)$. Given any such $\boldsymbol{y}$, because $\sum_j \delta_j f(\boldsymbol{z}_j) = f(\sum_j \delta_j \boldsymbol{z}_j)$, there must exist $\boldsymbol{z}_j \in \mathrm{conv}(X)$ such that $\boldsymbol{y} = \sum_j \delta_j f(\boldsymbol{z}_j)$. Let $\boldsymbol{z}_j = \sum_i \theta_i \boldsymbol{x}_i$ where $\sum_i \theta_i = 1$ and $\boldsymbol{x}_i \in X$ for all i. Then, we have

$$\boldsymbol{y} = \sum_j \delta_j f(\boldsymbol{z}_j) = \sum_j \delta_j f\big(\sum_i \theta_i \boldsymbol{x}_i\big) \tag{31}$$

$$\in \sum_j \delta_j f(\mathrm{conv}(X)) \tag{32}$$

$$\subseteq \mathrm{conv}(f(\mathrm{conv}(X))) = f(\mathrm{conv}(X)) \tag{33}$$

where the last equality is due to the fact that f is affine and thus convex.

So, for any $\boldsymbol{y} \in \mathrm{conv}(f(X))$, we always have $\boldsymbol{y} \in f(\mathrm{conv}(X))$, which yields $\mathrm{conv}(f(X)) \subseteq f(\mathrm{conv}(X))$. This completes the proof.

APPENDIX: PROOF OF THEOREM 2

The proof of Theorem 2 is similar to that of Theorem 1. We first investigate the DoF region outer bound with respect to $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k})$, followed by the achievability with respect to $(\{d_k^{[m]}\}_{k,m})$, and finally we prove the optimality by connecting those two. In particular, for the converse, we replace the DoF region outer bound in Lemmas 2 and 3 by the following one.

Lemma 11. *For the M^K regular TIM instances, the achievable DoF tuple $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k}) \in \mathbb{R}_+^{MK}$ should satisfy*

$$d_{k,\mathrm{sum}}^{[m_k]} \le 1, \quad \forall k \in [K],\ m_k \in [M] \tag{34a}$$

$$d_{k,\mathrm{sum}}^{[m_k]} + d_{j,\mathrm{sum}}^{[m_j]} \le 1, \quad \forall\ k,j \in [K],\ m_k, m_j \in [M], \quad \text{s.t. } (v_k^{[m_k]}, v_j^{[m_j]}) \in E \tag{34b}$$

where E is the edge set of the multi-state conflict graph $U(D)$.

Proof. The proof of this lemma is similar to that of Lemma 3. Every edge presented in every single-state conflict graph will be an edge in the multi-state conflict graph. As such, the collection of all edge inequalities from all single-state conflict graphs $\{U(D^{[\tilde{\boldsymbol{m}}]}), \forall \tilde{\boldsymbol{m}} \in [M]^K\}$ can be collected directly from the multi-state conflict graph $U(D)$. □

The polytope defined by the edge inequalities in (34) is usually referred to as the fractional stable set polytope. In particular, the DoF region outer bound in (34) is the fractional stable set polytope of $U(D)$ with respect to the DoF tuple $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k})$. Similarly, the DoF region outer bounds in (15) and (34) are identical, and can be represented as the convex hull of its extreme points, i.e., $\mathcal{P}^* = \mathcal{P}' = \mathrm{conv}(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{k,m_k})$, where $(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{k,m_k})$ denotes the extreme points.

For the achievability, we first introduce the following result in polyhedral combinatorics.

Lemma 12. *(From [20, Proposition 2.1] [21, Theorem 64.7]) The fractional stable set polytope is half-integral, that is, the coordinates of the extreme points can only be $\{0, \frac{1}{2}, 1\}$.*

By the above lemma, we conclude that the extreme points $(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{k,m_k})$ of the DoF region outer bound in (34) can only take values from $\{0, \frac{1}{2}, 1\}$. Then, we proceed to an analogous result to Lemma 7.

Lemma 13. *Given* (17), *if $\{\mathcal{T}_k^{[m]}\}_m$ is totally ordered, the half-integral-valued DoF tuple $(\{d_{k,\mathrm{sum}}^{[m_k]}\}_{k,m_k}) \in \mathbb{R}_+^{MK}$ leads to the corresponding half-integral-valued $(\{d_k^{[m]}\}_{k,m}) \in \mathbb{R}_+^{MK}$.*

Proof. First, given any extreme point $(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{k,m_k})$ of the DoF region defined by (34), if $\mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}$ we have $d_{k,\mathrm{sum}}^{*[m_1]} \ge d_{k,\mathrm{sum}}^{*[m_2]}$. This can be proved in a similar way as that of Lemma 10. It is still due to the **Property** for the multi-state conflict graph, where the clique Q reduces an edge. As such, if $\mathcal{T}_k^{[m_1]} \subseteq \mathcal{T}_k^{[m_2]}$, for the edge inequality involving $d_{k,\mathrm{sum}}^{[m_1]}$, there must exist another edge inequality by replacing $d_{k,\mathrm{sum}}^{[m_1]}$ with $d_{k,\mathrm{sum}}^{[m_2]}$.

Further, we arrange the relation between $(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{m_k})$ and $(\{d_k^{[m]}\}_m)$ in a system of linear equations, where $(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{m_k})$ is in an increasing order. One-step Gaussian elimination in a predetermined order will preserve the half-integrality, because the difference of two increasingly ordered values in half-integral $(\{d_{k,\mathrm{sum}}^{*[m_k]}\}_{m_k})$ still falls in $\{0, \frac{1}{2}, 1\}$. Thus, the lemma follows. □

Lemma 13 indicates that, if DoF tuples of the extreme points $(\{d_{k,\mathrm{sum}}^{[*m_k]}\}_{k,m_k})$ are half-integral, the corresponding DoF tuples $(\{d_k^{[m]}\}_{k,m})$ are also half-integral. For the achievability, we aim to prove the half-integral DoF tuples $(\{d_k^{[m]}\}_{k,m})$ are achievable by interference alignment.

Before proceeding further, we introduce the linear precoding at the transmitters and receivers as the encoding and decoding functions respectively. Specifically, the transmitted signal at Transmitter i over T channel uses is produced by

$$\boldsymbol{X}_i = \sum_{m=1}^{M} \boldsymbol{V}_i^{[m]} X_i^{[m]}, \quad \forall i \tag{35}$$

where the scalar symbol $X_i^{[m]}$ comes from the codeword encoded from the message $W_i^{[m]}$ using an independent Gaussian codebook,[2] and all symbols are jointly superposed with each aligned to the subspace spanned by $\boldsymbol{V}_i^{[m]} \in \mathbb{C}^{T\times 1}$. The precoding matrix $\boldsymbol{V}_i^{[m]}$ depends only on the network topologies $\{G^{[m]}\}_m$ over M states, and T will be determined by design.

The received signal at Receiver k at the m'-th state over T channel use is thus given by

$$\boldsymbol{Y}_k^{[m']} = \sum_{i\in\mathcal{T}_k} h_{ki}^{[m']}\boldsymbol{X}_i + \boldsymbol{Z}_k^{[m']}, \tag{36}$$

$$= \sum_{i\in\mathcal{T}_k}\sum_{m=1}^{M} h_{ki}^{[m']}\boldsymbol{V}_i^{[m]}X_i^{[m]} + \boldsymbol{Z}_k^{[m']} \tag{37}$$

In general, opportunistic TIM at the receiver side is a subspace decoding rule that opportunistically extracts information from subspace. In particular, a linear precoding matrix $\boldsymbol{U}_k^{[m']} \in \mathbb{C}^{\pi_k(m)\times T}$ is applied to $\boldsymbol{Y}_k^{[m']}$ at State m' for Receiver k to opportunistically decode desired basic and opportunistic messages.[3] After applying the precoding matrix $\boldsymbol{U}_k^{[m']}$ at State m', we have

$$\boldsymbol{U}_k^{[m']}\boldsymbol{Y}_k^{[m']} \tag{38}$$

$$= \sum_{i\in\mathcal{T}_k}\sum_{m=1}^{M} h_{ki}^{[m']}\boldsymbol{U}_k^{[m']}\boldsymbol{V}_i^{[m]}X_i^{[m]} + \boldsymbol{U}_k^{[m']}\boldsymbol{Z}_k^{[m']} \tag{39}$$

$$= \sum_{m=1}^{\pi_k(m')} h_{kk}^{[m']}\boldsymbol{U}_k^{[m']}\boldsymbol{V}_k^{[m]}X_k^{[m]} \tag{40}$$

$$+ \sum_{m=\pi_k(m')+1}^{M} h_{kk}^{[m']}\boldsymbol{U}_k^{[m']}\boldsymbol{V}_k^{[m]}X_k^{[m]}$$

$$+ \sum_{i\in\mathcal{T}_k\backslash k}\sum_{m=1}^{M} h_{ki}^{[m']}\boldsymbol{U}_k^{[m']}\boldsymbol{V}_i^{[m]}X_i^{[m]} + \boldsymbol{U}_k^{[m']}\boldsymbol{Z}_k^{[m']} \tag{41}$$

and the successful decoding under the TIM setting satisfies the following conditions:

$$\det\Big(\sum_{m=1}^{\pi_k(m')}\boldsymbol{U}_k^{[m']}\boldsymbol{V}_k^{[m]}\Big) \neq 0, \quad \forall k \tag{42a}$$

$$\sum_{m=\pi_k(m')+1}^{M}\boldsymbol{U}_k^{[m']}\boldsymbol{V}_k^{[m]} = \boldsymbol{0}, \quad \forall k \tag{42b}$$

$$\sum_{m=1}^{M}\boldsymbol{U}_k^{[m']}\boldsymbol{V}_i^{[m]} = \boldsymbol{0}, \quad \forall i\in\mathcal{T}_k\backslash k, \quad \forall k \tag{42c}$$

where (42a) ensures that the desired messages $\{W_k^{[m]}\}_{m=1}^{\pi_k(m')}$ at State m' are recoverable, (42b) ensures the interference caused by opportunistic messages desired by higher order states is eliminated, and (42c) enforces the interference caused by other connected transmitters is also eliminated. These conditions yield the achievable DoF

$$d_k^{[m]} = \frac{1}{T}, \quad \forall m \leq \pi_k(m'). \tag{43}$$

Lemma 14. *If all the original and auxiliary network topologies are half-rate feasible, interference alignment achieves all the half-integral extreme points.*

Proof. Given a half-integral DoF tuple $(\{d_k^{[m]}\}_{k,m})$, we rearrange it as a $K\times M$ matrix $\boldsymbol{D}$ with $[\boldsymbol{D}]_{km} = d_k^{[m]}$. Regarding this matrix, we have the following observations with respect to the multi-state conflict digraph:

- There should exist at most one '1' or at most two '$\frac{1}{2}$' in each row of $\boldsymbol{D}$. The total sum of each row, corresponding to the sum DoF for a receiver, can only be 0, $\frac{1}{2}$, or 1.
- If $d_k^{[m]} = 1$, there should neither exist any transmitter carrying non-zero DoF messages interfering Receiver k, nor any receiver interfered by Transmitter k can decode any non-zero DoF messages. In other words, for any State m' such that $\pi_k(m') \geq m$, the corresponding vertex $v_k^{[m']}$ will not conflict with any other vertices carrying non-zero DoF messages in $U(D)$.
- If $d_k^{[m]} = \frac{1}{2}$, for Receiver k at State m' with $m \leq \pi_k(m')$, there are two cases.
  - There are no other non-zero DoF messages desired at State m': In this case, there should be at least one incoming arc from a vertex carrying non-zero DoF messages to $v_k^{[m']}$. If there is more than one interference, these vertices should belong to a single alignment set. It is because, according to [1], if the network is half-rate feasible, then there does not exist any internal conflict, so that any two interfering sources that can be aligned, should be aligned.
  - There exists another $\tilde{m} \leq \pi_k(m')$ such that $d_k^{[\tilde{m}]} = \frac{1}{2}$: In this case, Receiver k should neither be interfered by other transmitters carrying non-zero DoF messages, nor any receiver interfered by Transmitter k can decode any non-zero DoF messages. Similarly, $v_k^{[m']}$ will not conflict with other vertices with non-zero DoF messages in $U(D)$.

Based on the above observations, we assign a randomly generated vectors $\boldsymbol{V}_k^{[m]} \in \mathbb{C}^{2\times 1}$ to the message $W_k^{[m]}$ in multi-state conflict digraphs. For $d_k^{[m]} = 0$, we set $\boldsymbol{V}_k^{[m]} = \boldsymbol{0}$ if $m \leq \max_{m'} \pi_k(m')$; otherwise $\boldsymbol{V}_k^{[m]} = \emptyset$ because it is not a valid message. The vector assignment for the non-zero DoF messages obeys the following rules:

- If there exists $\tilde{m}$ such that $d_k^{[\tilde{m}]} = 1$, assign two linearly independent vectors to $W_k^{[\tilde{m}]}$ associated to two symbols.
- If there exist $m_1 \neq m_2$ such that $d_k^{[m_1]} = d_k^{[m_2]} = \frac{1}{2}$, assign two linearly independent vectors $\boldsymbol{V}_k^{[m_1]}$ and $\boldsymbol{V}_k^{[m_2]}$ to $W_k^{[m_1]}$ and $W_k^{[m_2]}$, respectively.

[2] In this work, the scalar symbol of each coded message is sufficient for the achievability proof. For simplicity, we sometime abuse the messages $W_i^{[m]}$ to represent the coded symbols $X_i^{[m]}$.

[3] Orthogonal access (e.g., TDMA/FDMA) is a special case of such a design, in which the precoding matrices $\boldsymbol{V}_i^{[m]}$ and $\boldsymbol{U}_k^{[m']}$ turn to be columns of identity matrix that indicate the on/off pattern of transmitters and receivers over time/frequency.

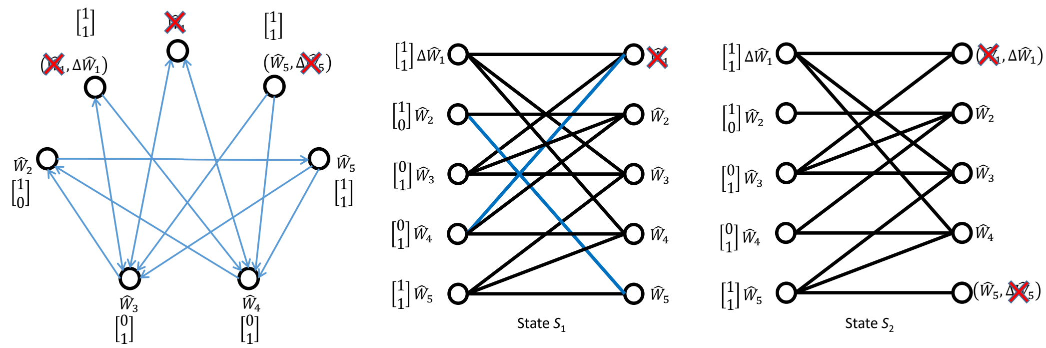

Fig. 6: The vector assignment on the multi-state conflict digraphs (Left), and the illustration of the transmission scheme over two states (Right) to achieve the extreme point $(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$ for Example 3.

- If there exists exactly one m_1 such that $d_k^{[m_1]}=\frac{1}{2}$, and $v_k^{[m_1]}$ falls into the same assignment set as $v_j^{[m_2]}$ where $d_j^{[m_2]}=\frac{1}{2}$, then assign the same vector $\boldsymbol{V}_k^{[m_1]}=\boldsymbol{V}_j^{[m_2]}$. Because there does not exist any internal alignment edge in its conflict graph $U(D)$, such an assignment will not introduce conflicts between the messages associated with $v_k^{[m_1]}$ and $v_j^{[m_2]}$.

At the receiver side, we check the decodability by designing receiver precoding matrices $\boldsymbol{U}_k^{[m']}$ for Receiver k at State m', aiming to decode all messages with DoF 1 and $\frac{1}{2}$.

When $d_k^{[\tilde{m}]}=1$ or $d_k^{[m_1]}=d_k^{[m_2]}=\frac{1}{2}$, Receiver k should not see any interference at State m' and two symbols are decodable. If there is only $d_k^{[m_1]}=\frac{1}{2}$, there are two cases.

- There is only one interference, say, from the message $W_j^{[\tilde{m}]}$. In this case, $\boldsymbol{U}_k^{[m']}=\boldsymbol{V}_j^{[\tilde{m}]\perp}$.
- There are multiple interfering messages from a single alignment set. In this case, these interferences should be perfectly aligned and occupy one-dimensional subspace, say $\boldsymbol{V}_j^{[\tilde{m}]}$, leaving one-dimensional interference-free subspace to the desired symbol. So, we have $\boldsymbol{U}_k^{[m']}=\boldsymbol{V}_j^{[\tilde{m}]\perp}$, by which one symbol is decodable within two-dimensional subspace, yielding DoF $\frac{1}{2}$.

Thus, the half-integral DoF tuple $(\{d_k^{[m]}\}_{k,m})$ can be achieved by carefully designing transmit and receive precoding matrices $\{\boldsymbol{V}_k^{[m]}\}_{k,m}$ and $\{\boldsymbol{U}_k^{[m']}\}_{k,m'}$. □

Example 3. Let us take a possible design of Example 2 for illustration. Given the DoF region by the constraints in (16), the nontrivial extreme points of the polytope are $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0)$, $(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$, $(1,1,0,0,0,0,1)$, and $(0,1,0,0,0,1,1)$. For instance, to achieve the extreme point $(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$, the transmit precoding matrices can be designed as follows:

$$\boldsymbol{V}_1^{[1]}=\begin{bmatrix}0\\0\end{bmatrix},\boldsymbol{V}_1^{[2]}=\begin{bmatrix}1\\1\end{bmatrix}, \tag{44}$$

$$\boldsymbol{V}_2^{[1]}=\begin{bmatrix}1\\0\end{bmatrix},\boldsymbol{V}_2^{[2]}=\emptyset, \tag{45}$$

$$\boldsymbol{V}_3^{[1]}=\begin{bmatrix}0\\1\end{bmatrix},\boldsymbol{V}_3^{[2]}=\emptyset, \tag{46}$$

$$\boldsymbol{V}_4^{[1]}=\begin{bmatrix}0\\1\end{bmatrix},\boldsymbol{V}_4^{[2]}=\emptyset, \tag{47}$$

$$\boldsymbol{V}_5^{[1]}=\begin{bmatrix}1\\1\end{bmatrix},\boldsymbol{V}_5^{[2]}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{48}$$

where the messages $\{W_3^{[1]},W_4^{[1]}\}$ are aligned, so are $\{W_1^{[2]},W_5^{[1]}\}$. The vector assignment is shown in Fig. 6.

Thus, the receive precoders can be designed as below:

$$\boldsymbol{U}_1^{[1]}=\begin{bmatrix}0\\0\end{bmatrix}^\intercal,\boldsymbol{U}_1^{[2]}=\begin{bmatrix}0&0\\1&0\end{bmatrix}, \tag{49}$$

$$\boldsymbol{U}_2^{[1]}=\begin{bmatrix}1\\0\end{bmatrix}^\intercal,\boldsymbol{U}_2^{[2]}=\begin{bmatrix}1\\0\end{bmatrix}^\intercal, \tag{50}$$

$$\boldsymbol{U}_3^{[1]}=\begin{bmatrix}1\\-1\end{bmatrix}^\intercal,\boldsymbol{U}_3^{[2]}=\begin{bmatrix}1\\-1\end{bmatrix}^\intercal, \tag{51}$$

$$\boldsymbol{U}_4^{[1]}=\begin{bmatrix}1\\-1\end{bmatrix}^\intercal,\boldsymbol{U}_4^{[2]}=\begin{bmatrix}1\\-1\end{bmatrix}^\intercal, \tag{52}$$

$$\boldsymbol{U}_5^{[1]}=\begin{bmatrix}0\\1\end{bmatrix}^\intercal,\boldsymbol{U}_5^{[2]}=\begin{bmatrix}0&1\\0&0\end{bmatrix}. \tag{53}$$

where for $k=2,3,4$, $\boldsymbol{U}_k^{[1]}=\boldsymbol{U}_k^{[2]}$ indicates that the same basic messages are desired over two states, and in $\boldsymbol{U}_1^{[2]}$ and $\boldsymbol{U}_5^{[2]}$, two rows are dedicated to the basic and opportunistic messages respectively.

To sum up, given the half-integrality of extreme points $d_{k,\text{sum}}^{*[m_k]}$ of $\mathcal{P}'$, and the half-integrality of corresponding $\{d_k^{[m]}\}_{k,m}$, we can design interference alignment schemes to achieve these half-integral DoF tuples $\{d_k^{[m]}\}_{k,m}$. Using time sharing we generate an achievable DoF region $\mathcal{P}=f(\text{conv}(\{d_k^{[m]}\}_{k,m}))$, which is no smaller than $\mathcal{P}'=\text{conv}(\{d_{k,\text{sum}}^{*[m_k]}\}_{k,m_k})$, i.e., $\mathcal{P}'\subseteq\mathcal{P}$. Finally, the optimality follows because on the other hand $\mathcal{P}\subseteq\mathcal{P}^*=\mathcal{P}'$. This completes the proof of Theorem 2.

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